Structure-preserving nonlinear model reduction for finite-volume models

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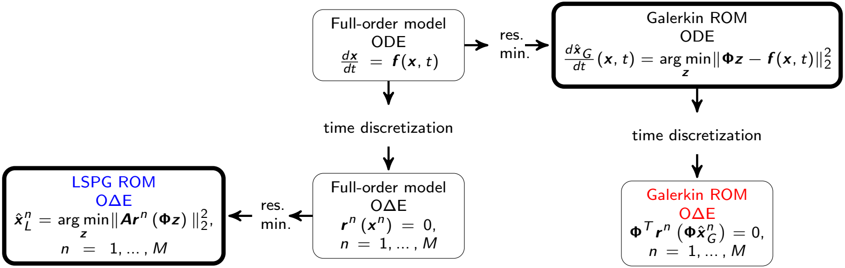
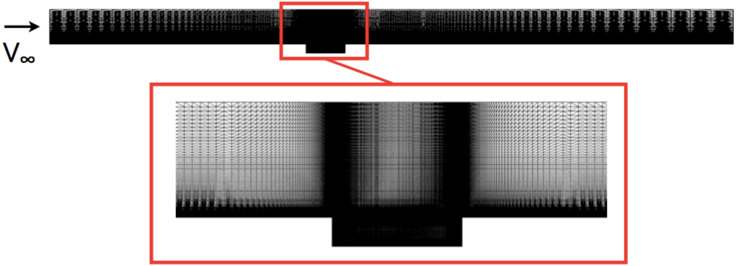
Optimize then discretize, or discretize then optimize?



- Galerkin: continuous-residual minimization
- LSPG [C. et al., 2011]: discrete-residual minimization

Comparative analysis: K. Carlberg, M. Barone, H. Antil, "Galerkin v. least-squares Petrov–Galerkin projection in nonlinear model reduction," Journal of Computational Physics, 330:693–734, 2017.

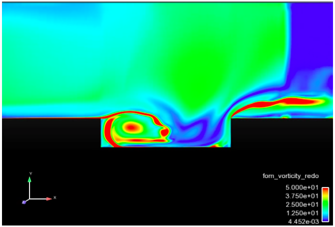
Cavity-flow problem Collaborator: M. Barone (SNL)



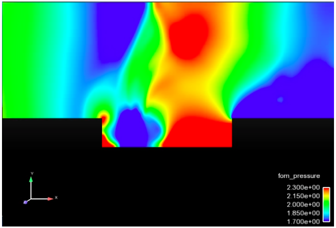
- Unsteady, compressible Navier–Stokes
- DES turbulence model
- Finite-volume discretization
- BDF2 linear multistep time integrator

- $M_\infty = 0.6$
- $\mathrm{Re} = 6.3 \times 10^6$
- 1.2×10^6 degrees of freedom
- CFD code: AERO-F [Farhat et al., 2003]

Full-order model responses



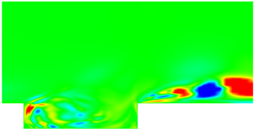
vorticity field



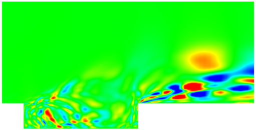
pressure field

POD modes $\mathbf{\Phi}$ (energy component)

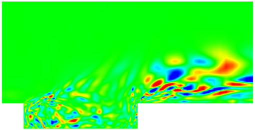
(a) mode 1



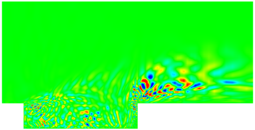
(b) mode 21



(c) mode 101

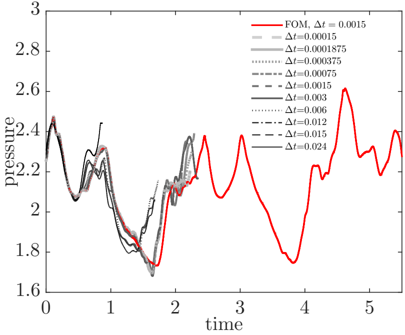


(d) mode 201

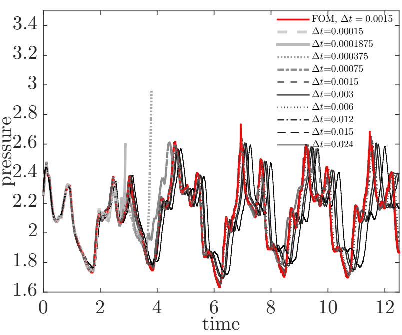


(e) mode 401

Galerkin and LSPG responses for basis dimension $p = 564$



(f) Galerkin



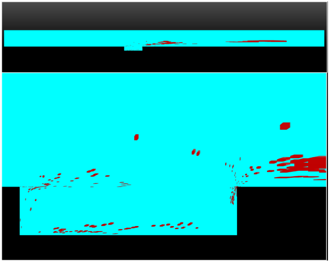
(g) LSPG

- Galerkin ROMs produce non-physical solutions
- LSPG ROMs
  - \+ accurate and stable (most time steps)
  - \- more expensive than the FOM (1.3 hours>1 hour, 48 CPU)

Sample mesh [C. et al., 2013]

$$\hat{\boldsymbol{x}}^n = \arg\min_{\hat{\boldsymbol{z}}\in\mathbb{R}^p} \| \underbrace{(\boldsymbol{P}\boldsymbol{\Phi}_R)^+ \boldsymbol{P}}_{\boldsymbol{A}_{\text{GNAT}}} \boldsymbol{r}^n(\boldsymbol{\Phi}\hat{\boldsymbol{z}}) \|_2^2$$

- $\boldsymbol{A}_{\text{GNAT}}$: gappy POD [Everson and Sirovich, 1995] approx of residual
- *Sample mesh*: Extract mesh subset needed to compute $\boldsymbol{P}\boldsymbol{r}^n$
  - *Related*: RID [Ryckelynck, 2005], subgrid [Haasdonk et al., 2008]



- Sample mesh: 4.1% nodes, 3.0% cells

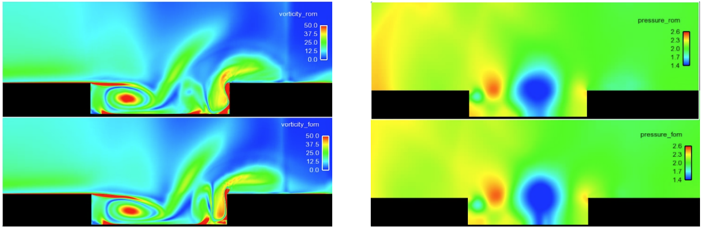
\+ Small problem size: can run on many fewer cores

GNAT performance ($t \leq 12.5$ sec)

vorticity field | pressure field

GNAT ROM

FOM



+ $< 1\%$ error in time-averaged drag
+ 229x CPU-hour savings
  - FOM: 5 hour x 48 CPU
  - GNAT ROM: 32 min x 2 CPU

Why is LSPG more accurate than Galerkin? [C. et al., 2017]

Theorem (Local *a posteriori* bounds: BDF schemes)

If the following conditions hold:

1. $\exists \kappa > 0$ *such that* $\|\boldsymbol{f}(\boldsymbol{x}, \cdot) - \boldsymbol{f}(\boldsymbol{y}, \cdot)\|_2 \leq \kappa \|\boldsymbol{x} - \boldsymbol{y}\|_2$, $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$
2. Δt *small enough such that* $0 < h := |\alpha_0| - |\beta_0| \kappa \Delta t$
3. *A BDF scheme is employed for time integration, then*

$$\|\delta \boldsymbol{x}_G^n\|_2 \leq \frac{1}{h} \|\boldsymbol{r}_G^n(\boldsymbol{\Phi} \hat{\boldsymbol{x}}_G^n)\|_2 + \frac{1}{h} \sum_{\ell=1}^{k} |\alpha_\ell| \|\delta \boldsymbol{x}_G^{n-\ell}\|_2$$

$$\|\delta \boldsymbol{x}_L^n\|_2 \leq \frac{1}{h} \min_{\boldsymbol{y} \in \mathrm{Ran}(\boldsymbol{\Phi})} \|\boldsymbol{r}_P^n(\boldsymbol{y})\|_2 + \frac{1}{h} \sum_{\ell=1}^{k} |\alpha_\ell| \|\delta \boldsymbol{x}_L^{n-\ell}\|_2$$

- $\delta \boldsymbol{x}_G^n := \boldsymbol{x}_\star^n - \boldsymbol{\Phi} \hat{\boldsymbol{x}}_G^n$.
- $\delta \boldsymbol{x}_L^n := \boldsymbol{x}_\star^n - \boldsymbol{\Phi} \hat{\boldsymbol{x}}_L^n$

LSPG minimizes the error bound sequentially in time!

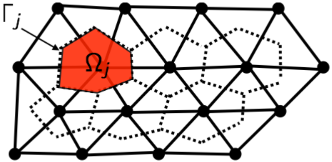
Structure preservation in model reduction

- Stability [Moore, 1981, Bond and Daniel, 2008, Amsallem and Farhat, 2012, Kalashnikova et al., 2014]
- Second order [Freund, 2005, Salimbahrami, 2005, Chahlaoui, 2015]
- Delay
  [Beattie and Gugercin, 2008, Michiels et al., 2011, Schulze and Unger, 2015]
- Bilinear [Zhang and Lam, 2002, Benner and Damm, 2011, Benner and Breiten, 2012, Flagg and Gugercin, 2015]
- Inf-sup stability [Rozza and Veroy, 2007, Gerner and Veroy, 2012, Rozza et al., 2013, Ballarin et al., 2014]
- Passivity [Phillips et al., 2003, Sorensen, 2005, Wolf et al., 2010]
- Energy conservation
  [An et al., 2008, Farhat et al., 2014, Farhat et al., 2015]
- Lagrangian structure [Lall et al., 2003, C. et al., 2015]
- (Port-)Hamiltonian [Polyuga and van der Schaft, 2008, Beattie and Gugercin, 2011, Afkham and Hesthaven, 2016, Chaturantabut et al., 2016, Peng and Mohseni, 2016]

What structure should we preserve in finite-volume models?

Finite-volume discretization: full-order model

Full-order model ODE: $\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, t)$



$$x_{\mathcal{I}(i,j)} = \frac{1}{|\Omega_j|}\int_{\Omega_j} u_i(\vec{x}, t)d\vec{x}$$
$$f_{\mathcal{I}(i,j)} = -\frac{1}{|\Omega_j|}\int_{\Gamma_j} u_i(\vec{x}, t)v_\ell(\vec{x}, t)n_\ell(\vec{x})d\vec{x}$$

- Conserved variables u_i, $i = 1, \ldots, n_u$
- ℓ-component of velocity v_ℓ, $\ell = 1, \ldots, d$ with $d \in \{1, 2, 3\}$
- ℓ-component of normal n_ℓ, $\ell = 1, \ldots, d$
- $\mathcal{I} : \{1, \ldots, n_u\} \times \{1, \ldots, N_\Omega\} \to \{1, \ldots, N\}$, $N = n_u N_\Omega$

Full-order model OΔE: $\boldsymbol{r}^n(\boldsymbol{x}^n) = 0$, $n = 1, \ldots, M$

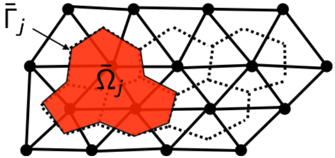
$$r^n_{\mathcal{I}(i,j)} = \frac{1}{|\Omega_j|}\int_{\Omega_j} u_i(\vec{x}, t^{n+1}) - u_i(\vec{x}, t^n)d\vec{x} + \frac{1}{|\Omega_j|}\int_{t^n}^{t^{n+1}}\int_{\Gamma_j} u_i(\vec{x}, t)v_\ell(\vec{x}, t)n_\ell(\vec{x})d\vec{x}dt$$

$r^n_{\mathcal{I}(i,j)}$: violation of conservation in u_i over Ω_j and $[t^n, t^{n+1}]$.

Finite-volume discretization: LSPG ROM

LSPG ROM: $\underset{\boldsymbol{z}}{\text{minimize}} \; \|\boldsymbol{A}\boldsymbol{r}^n(\boldsymbol{\Phi}\boldsymbol{z})\|_2^2$

+ Minimizes (weighted) sum of squares of conservation-law violations
- Does not ensure conservation anywhere!



LSPG-FV ROM:

$$\underset{\boldsymbol{z}}{\text{minimize}} \; \|\boldsymbol{A}\boldsymbol{r}^n(\boldsymbol{\Phi}\boldsymbol{z})\|_2^2$$
$$\text{subject to } \bar{\boldsymbol{r}}^n(\boldsymbol{\Phi}\boldsymbol{z}) = \boldsymbol{0}$$

$$\bar{r}^n_{\bar{\mathcal{I}}(i,j)} = \frac{1}{|\bar{\Omega}_j|}\int_{\bar{\Omega}_j} u_i(\vec{x}, t^{n+1}) - u_i(\vec{x}, t^n) d\vec{x} + \frac{1}{|\bar{\Omega}_j|}\int_{t^n}^{t^{n+1}}\int_{\bar{\Gamma}_j} u_i(\vec{x}, t) v_\ell(\vec{x}, t) n_\ell(\vec{x}) d\vec{x} dt$$

- $\bar{\mathcal{I}} : \{1, \ldots, n_u\} \times \{1, \ldots, N_{\bar{\Omega}}\} \to \{1, \ldots, \bar{N}\}$, with $\bar{N} = n_u N_{\bar{\Omega}}$

+ Minimizes sum of squares of conservation-law violations
+ Ensure conservation laws are enforced over $N_{\bar{\Omega}}$ subdomains

LSPG-FV: three cases

- $\boldsymbol{r}^n : \mathbb{R}^N \rightarrow \mathbb{R}^N$
- $\boldsymbol{\Phi} \in \mathbb{R}_\star^{N \times p}$
- $\bar{\boldsymbol{r}}^n : \mathbb{R}^p \rightarrow \mathbb{R}^{\bar{N}}$

1. Underdetermined constraint problem ($p > \bar{N}$)

$$\underset{\boldsymbol{z}}{\text{minimize}} \; \|\boldsymbol{A}\boldsymbol{r}^n(\boldsymbol{\Phi}\boldsymbol{z})\|_2^2$$
$$\text{subject to } \bar{\boldsymbol{r}}^n(\boldsymbol{\Phi}\boldsymbol{z}) = \mathbf{0}$$

   - Solve with sequential quadratic programming (SQP)
   - \+ Conservation over $\bar{\Omega}_j$, $j = 1, \ldots, N_{\bar{\Omega}}$ ensured

2. Well-posed constraint problem ($p = \bar{N}$)

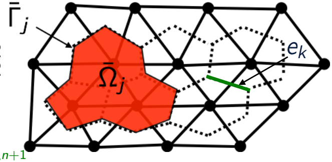
$$\bar{\boldsymbol{r}}^n(\boldsymbol{\Phi}\boldsymbol{z}) = \mathbf{0}$$

3. Overdetermined constraint problem ($p < \bar{N}$)

$$\underset{\boldsymbol{z}}{\text{minimize}} \; \|\boldsymbol{A}\boldsymbol{r}^n(\boldsymbol{\Phi}\boldsymbol{z})\|_2^2 + \mu\|\bar{\boldsymbol{A}}\bar{\boldsymbol{r}}^n(\boldsymbol{\Phi}\boldsymbol{z})\|_2^2$$

   - Penalty parameter $\mu \in \mathbb{R}_+$
   - \- No conservation guaranteed

Hyper-reduction



GNAT-FV ROM: $\underset{z}{\text{min.}} \; \| \underbrace{(\boldsymbol{P\Phi}_R)^+ \boldsymbol{P}}_{\boldsymbol{A}_{\text{GNAT}}} \boldsymbol{r}^n(\boldsymbol{\Phi z}) \|_2^2$

subject to $\bar{\boldsymbol{r}}^n(\boldsymbol{\Phi z}) = \boldsymbol{0}$

$$\bar{r}^n_{\bar{\mathcal{I}}(i,j)} = \frac{1}{|\bar{\Omega}_j|} \int_{\bar{\Omega}_j} u_i(\vec{x}, t^{n+1}) - u_i(\vec{x}, t^n) d\vec{x} + \frac{1}{|\bar{\Omega}_j|} \int_{t^n}^{t^{n+1}} \underbrace{\int_{\bar{\Gamma}_j} u_i(\vec{x}, t) v_\ell(\vec{x}, t) n_\ell(\vec{x}) d\vec{x}\, dt}_{\text{nonlinear flux}}$$

$$= \underbrace{\boldsymbol{a}(i,j)^T \boldsymbol{\Phi}}_{\text{linear (precompute)}} (\hat{\boldsymbol{x}}^{n+1} - \hat{\boldsymbol{x}}^n) + \frac{1}{|\bar{\Omega}_j|} \int_{t^n}^{t^{n+1}} \underbrace{\boldsymbol{b}(i,j)^T \boldsymbol{g}(\boldsymbol{x}(t))}_{\text{nonlinear}} dt$$

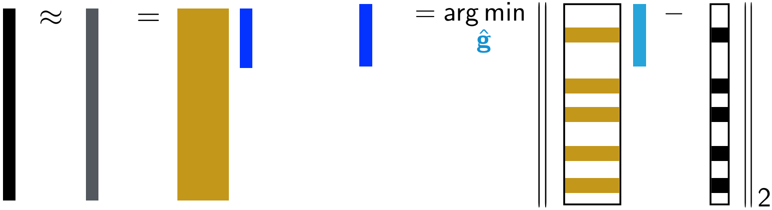
- Interface flux $g_{\mathcal{J}(i,k)} = \int_{e_k} u_i(\vec{x}, t) v_\ell(\vec{x}, t) \bar{n}_\ell(\vec{x}) d\vec{x}$
- $\mathcal{J} : \{1, \ldots, n_u\} \times \{1, \ldots, N_e\} \to \{1, \ldots, N_g\}$, with $N_g = n_u N_e$
- $\boldsymbol{a} : \{1, \ldots, n_u\} \times \{1, \ldots, N_{\bar{\Omega}}\} \to \mathbb{R}_+^N$
- $\boldsymbol{b} : \{1, \ldots, n_u\} \times \{1, \ldots, N_{\bar{\Omega}}\} \to \{-1, 0, 1\}^{N_g}$
- ℓ-component of edge normal $\bar{n}_\ell$, $\ell = 1, \ldots, d$

Approximate interface flux $\boldsymbol{g} : \mathbb{R}^N \to \mathbb{R}^{N_g}$ *using gappy POD.*

Hyper-reduction via Gappy POD [Everson and Sirovich, 1995]

- *Offline*. Compute:
  1. $\boldsymbol{\Phi}_{\boldsymbol{g}} \in \mathbb{R}_*^{N_g \times n_g}$ (POD)
  2. $\boldsymbol{P}_{\boldsymbol{g}} \in \{0,1\}^{n_s \times N_g}$ (sample-mesh edges)
- *Online*. Approximate flux via gappy POD:

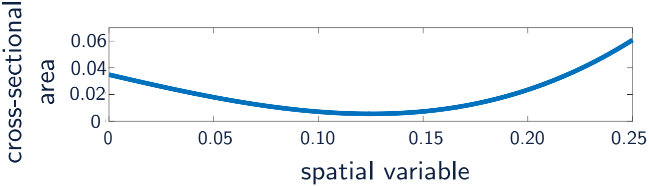
1. $\boldsymbol{g}(\boldsymbol{x}) \approx \tilde{g}(\boldsymbol{x}) = \boldsymbol{\Phi}_{\boldsymbol{g}}\hat{\boldsymbol{g}}(\boldsymbol{x})$ 2. $\hat{\boldsymbol{g}}(\boldsymbol{x}) = \arg\min_{\hat{\boldsymbol{g}}} \|\boldsymbol{P}_{\boldsymbol{g}}\boldsymbol{\Phi}_{\boldsymbol{g}}\hat{\boldsymbol{g}} - \boldsymbol{P}_{\boldsymbol{g}}\boldsymbol{g}(\boldsymbol{x})\|_2$



GNAT-FV ROM: $$\underset{\boldsymbol{z}}{\text{min.}} \quad \left\|(\boldsymbol{P}\boldsymbol{\Phi}_R)^+ \boldsymbol{P}\boldsymbol{r}^n(\boldsymbol{\Phi}\boldsymbol{z})\right\|_2^2$$
$$\text{subject to } \tilde{\boldsymbol{r}}^n(\boldsymbol{\Phi}\boldsymbol{z}) = \boldsymbol{0}$$

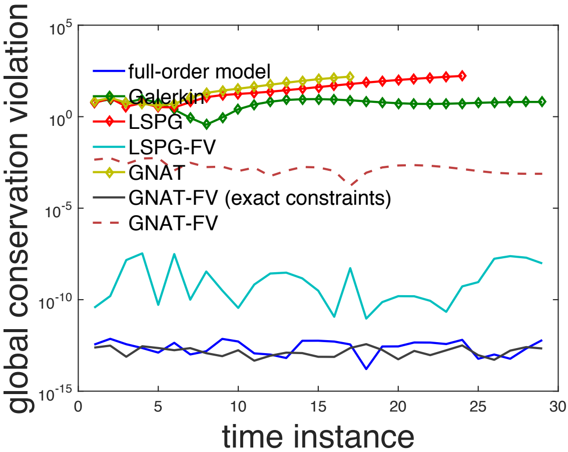
$$\tilde{r}^n_{\bar{\mathcal{I}}(i,j)} = \underbrace{\boldsymbol{a}(i,j)^T\boldsymbol{\Phi}}_{\text{linear (precompute)}}(\hat{\boldsymbol{x}}^{n+1} - \hat{\boldsymbol{x}}^n) + \frac{1}{|\bar{\Omega}_j|}\int_{t^n}^{t^{n+1}} \underbrace{\boldsymbol{b}(i,j)^T\boldsymbol{\Phi}_{\boldsymbol{g}}(\boldsymbol{P}_{\boldsymbol{g}}\boldsymbol{\Phi}_{\boldsymbol{g}})^+}_{\text{linear (precompute)}} \underbrace{\boldsymbol{P}_{\boldsymbol{g}}\boldsymbol{g}(\boldsymbol{x})}_{\text{sample flux}}\, dt$$

Example: Quasi-1D Euler equation



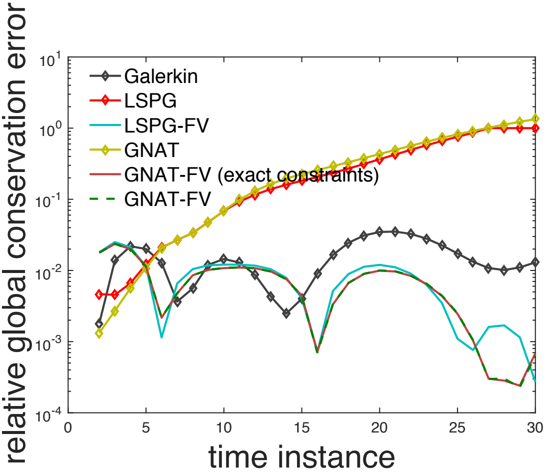
- $n_u = 3$ conserved quantities $\boldsymbol{u} = (\rho, \rho v, E)$
- Number of control volumes $N_\Omega = 100$
- Total time steps $M = 30$
- Time step $\Delta t = 0.01$
- Training: Mach number $M \in \{1.7, 1.8, 1.9, 2.0\}$
- Online: Mach number $M = 1.75$
- ROM parameters: $p = 5$, $n_g = 20$

Global conservation ($N_{\bar{\Omega}} = 1$)



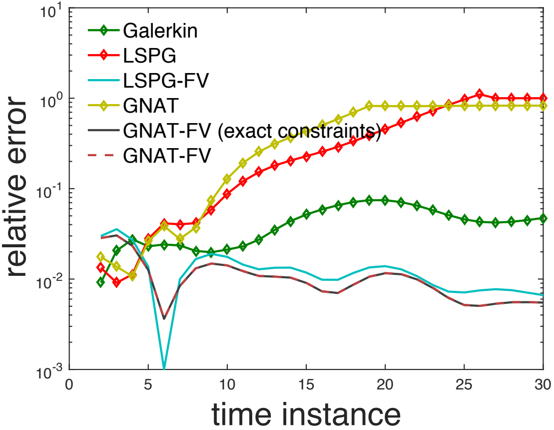
+ Global conservation exactly satisfied for exact constraints
+ GNAT-FV: accurate conservation-law approximation

Global conservation ($N_{\bar{\Omega}} = 1$)



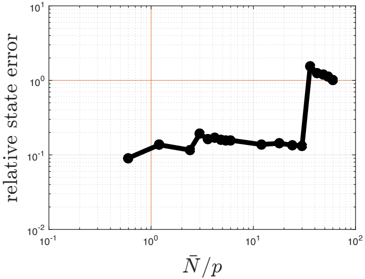
- Conservation-law constraints: small (but nonzero) error in globally conserved quantities

Global conservation ($N_{\bar{\Omega}} = 1$)

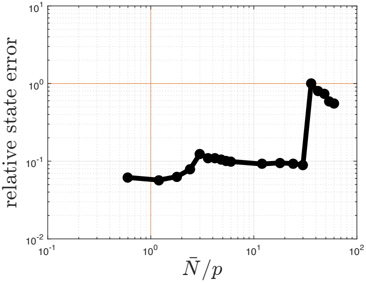


+ Relative state error roughly matches global conservation error

Varying number of subdomains $N_{\bar{\Omega}}$ (penalty parameter $\mu = 10^3$)



(h) LSPG-FV



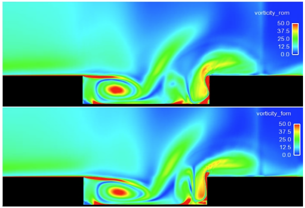
(i) GNAT-FV

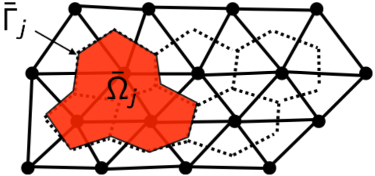
+ Conservation-law constraints reduce error by $10\times$
+ GNAT-FV approximately the same accuracy as LSPG-FV
+ Best accuracy for global conservation ($N_{\bar{\Omega}} = 1$)

Conclusions

- Structure-preserving model reduction for nonlinear finite-volume models
  - Conservation-law violation equality constraints
  - Enforces conservation over subdomains
  - Hyper-reduction by applying gappy POD flux approximation
- Numerical experiments
  + Constraints reduced both state and global conservation errors
  - Best results obtained for global conservation ($N_{\bar{\Omega}} = 1$)

Questions?





References:

- K. Carlberg, Y. Choi, S. Sargsyan. Structure-preserving model reduction for finite-volume models. *In preparation*, 2016.
- K. Carlberg, M. Barone, and H. Antil. Galerkin v. least-squares Petrov–Galerkin projection in nonlinear model reduction. Journal of Computational Physics, 330:693–734, 2017.
- K. Carlberg, C. Farhat, J. Cortial, and D. Amsallem. The GNAT method for nonlinear model reduction: effective implementation and application to computational fluid dynamics and turbulent flows. Journal of Computational Physics, 242:623–647, 2013.

Acknowledgments

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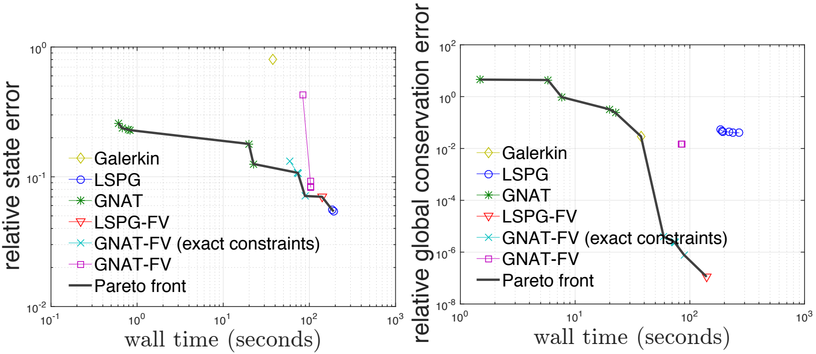

- GNAT: Pareto optimal for small wall times
+ GNAT-FV, LSPG-FV: Pareto optimal for smaller errors

Afkham, M. B. and Hesthaven, J. S. (2016).
Structure preserving model reduction of parametric hamiltonian systems.
Technical report.

Amsallem, D. and Farhat, C. (2012).
Stabilization of projection-based reduced-order models.
International Journal for Numerical Methods in Engineering, 91(4):358–377.

An, S., Kim, T., and James, D. (2008).
Optimizing cubature for efficient integration of subspace deformations.
ACM Transactions on Graphics (TOG), 27(5):165.

Ballarin, F., Manzoni, A., Quateroni, A., and Rozza, G. (2014).
Supremizer stabilization of pod-galerkin approximation of parametrized navier-stokes equations.

Int. J. Numer. Meth. Engng, Special Issue on Model Reduction.

Beattie, C. and Gugercin, S. (2011).
Structure-preserving model reduction for nonlinear port-Hamiltonian systems.
In Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on, pages 6564–6569. IEEE.

Beattie, C. A. and Gugercin, S. (2008).
Interpolation theory for structure-preserving model reduction.
In CDC, pages 4204–4208.

Benner, P. and Breiten, T. (2012).
Interpolation-based \calh_2-model reduction of bilinear control systems.
SIAM Journal on Matrix Analysis and Applications, 33(3):859–885.

Benner, P. and Damm, T. (2011).

Lyapunov equations, energy functionals, and model order reduction of bilinear and stochastic systems.
SIAM Journal on Control and Optimization, 49(2):686–711.

Bond, B. and Daniel, L. (2008).
Guaranteed stable projection-based model reduction for indefinite and unstable linear systems.
In Proceedings of the 2008 IEEE/ACM International Conference on Computer-Aided Design, pages 728–735. IEEE Press.

Bui-Thanh, T., Willcox, K., and Ghattas, O. (2008).
Model reduction for large-scale systems with high-dimensional parametric input space.
SIAM Journal on Scientific Computing, 30(6):3270–3288.

C., K., Barone, M., and Antil, H. (2017).
Galerkin v. least-squares Petrov–Galerkin projection in nonlinear model reduction.
Journal of Computational Physics, 330:693–734.

C., K., Bou-Mosleh, C., and Farhat, C. (2011).
Efficient non-linear model reduction via a least-squares Petrov–Galerkin projection and compressive tensor approximations.
International Journal for Numerical Methods in Engineering, 86(2):155–181.

C., K., Farhat, C., Cortial, J., and Amsallem, D. (2013).
The GNAT method for nonlinear model reduction: effective implementation and application to computational fluid dynamics and turbulent flows.
Journal of Computational Physics, 242:623–647.

C., K., Tuminaro, R., and Boggs, P. (2015).
Preserving Lagrangian structure in nonlinear model reduction with application to structural dynamics.
SIAM J. Sci. Comput., 37(2):B153—B184.

Chahlaoui, Y. (2015).

Recursive low-rank approximation and model reduction of second-order systems.
arXiv preprint arXiv:1511.09034.

Chaturantabut, S., Beattie, C., and Gugercin, S. (2016).
Structure-preserving model reduction for nonlinear port-hamiltonian systems.
arXiv preprint arXiv:1601.00527.

Everson, R. and Sirovich, L. (1995).
Karhunen–Loève procedure for gappy data.
Journal of the Optical Society of America A, 12(8):1657–1664.

Farhat, C., Avery, P., Chapman, T., and Cortial, J. (2014).
Dimensional reduction of nonlinear finite element dynamic models with finite rotations and energy-based mesh sampling and weighting for computational efficiency.
International Journal for Numerical Methods in Engineering, 98(9):625–662.

Farhat, C., Chapman, T., and Avery, P. (2015).

Structure-preserving, stability, and accuracy properties of the energy-conserving sampling and weighting method for the hyper reduction of nonlinear finite element dynamic models.
International Journal for Numerical Methods in Engineering, 102(5):1077–1110.

Farhat, C., Geuzaine, P., and Brown, G. (2003).
Application of a three-field nonlinear fluid-structure formulation to the prediction of the aeroelastic parameters of an F-16 fighter.
Computers & Fluids, 32(1):3–29.

Flagg, G. and Gugercin, S. (2015).
Multipoint volterra series interpolation and h_2 optimal model reduction of bilinear systems.
SIAM Journal on Matrix Analysis and Applications, 36(2):549–579.

Freund, R. W. (2005).

Padé-type model reduction of second-order and higher-order linear dynamical systems.
In Dimension Reduction of Large-Scale Systems, pages 191–223. Springer.

Gerner, A.-L. and Veroy, K. (2012).
Certified reduced basis methods for parametrized saddle point problems.
SIAM Journal on Scientific Computing, 34(5):A2812–A2836.

Haasdonk, B., Ohlberger, M., and Rozza, G. (2008).
A reduced basis method for evolution schemes with parameter-dependent explicit operators.
Electronic Transactions on Numerical Analysis, 32:145–161.

Kalashnikova, I., van Bloemen Waanders, B., Arunajatesan, S., and Barone, M. (2014).
Stabilization of projection-based reduced order models for linear time-invariant systems via optimization-based eigenvalue reassignment.

Computer Methods in Applied Mechanics and Engineering, 272:251–270.

Lall, S., Krysl, P., and Marsden, J. (2003).
Structure-preserving model reduction for mechanical systems.
Physica D: Nonlinear Phenomena, 184(1-4):304–318.

LeGresley, P. A. (2006).
Application of Proper Orthogonal Decomposition (POD) to Design Decomposition Methods.
PhD thesis, Stanford University.

Michiels, W., Jarlebring, E., and Meerbergen, K. (2011).
Krylov-based model order reduction of time-delay systems.
SIAM Journal on Matrix Analysis and Applications, 32(4):1399–1421.

Moore, B. (1981).
Principal component analysis in linear systems: Controllability, observability, and model reduction.
Automatic Control, IEEE Transactions on, 26(1):17–32.

Peng, L. and Mohseni, K. (2016).
Symplectic model reduction of hamiltonian systems.
SIAM Journal on Scientific Computing, 38(1):A1–A27.

Phillips, J. R., Daniel, L., and Silveira, L. M. (2003).
Guaranteed passive balancing transformations for model order reduction.
IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 22(8):1027–1041.

Polyuga, R. and van der Schaft, A. (2008).
Structure preserving model reduction of port-Hamiltonian systems.
In 18th International Symposium on Mathematical Theory of Networks and Systems, Blacksburg, Virginia, USA.

Rozza, G., Huynh, D. P., and Manzoni, A. (2013).
Reduced basis approximation and a posteriori error estimation for stokes flows in parametrized geometries: roles of the inf-sup stability constants.

Numerische Mathematik, 125(1):115–152.

Rozza, G. and Veroy, K. (2007).
On the stability of the reduced basis method for stokes equations in parametrized domains.
Computer methods in applied mechanics and engineering, 196(7):1244–1260.

Ryckelynck, D. (2005).
A priori hyperreduction method: an adaptive approach.
Journal of Computational Physics, 202(1):346–366.

Salimbahrami, S. B. (2005).
Structure preserving order reduction of large scale second order models.
PhD thesis, Universität München.

Schulze, P. and Unger, B. (2015).
Data-driven interpolation of dynamical systems with delay.

Sorensen, D. C. (2005).

Passivity preserving model reduction via interpolation of spectral zeros.
Systems & Control Letters, 54(4):347–360.

Wolf, T., Lohmann, B., Eid, R., and Kotyczka, P. (2010).
Passivity and structure preserving order reduction of linear port-hamiltonian systems using krylov subspaces.
European Journal of Control, 16(4):401–406.

Zahm, O. and Nouy, A. (2016).
Interpolation of inverse operators for preconditioning parameter-dependent equations.
SIAM Journal on Scientific Computing.

Zhang, L. and Lam, J. (2002).
On h 2 model reduction of bilinear systems.
Automatica, 38(2):205–216.